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Measuring real value and inflation

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MEASURING REAL VALUE AND INFLATION

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ABSTRACT

The most important economic measures are monetary. They have many different names, are derived in different theories and employ different formulas. Yet, they all attempt to do basically the same thing: to separate a change in nominal value into a 'real part' due to the changes in quantities and an inflation due to the changes of prices. Examples are: real national product and its components, the GNP deflator, the CPI, various measures related to consumer surplus, as well as the large number of formulas for price and quantity indexes that have been proposed.

The theories that have been developed to derive these measures are largely unsatisfactory. The axiomatic theory of indexes does not make clear which economic problem a particular formula can be used to solve. The economic theories are for the most part based on unrealistic assumption. For example, the theory of the CPI is usually developed for a single consumer with homothetic preferences and then applied to a large aggregate of diverse consumers with non-homothetic preferences.

In this paper I develop a unitary theory that can be used in all situations in which monetary measures have been used. The theory implies a uniquely optimal measure which turns out to be the Törnqvist index. I review, and partly re-interpret the derivations of this index in the literature and provide several new derivations.

The paper also covers several related topics, particularly the presently unsatisfactory determination of the components of real GDP.

1. INTRODUCTION

This paper offers unified theory that is applicable in all instances where economists have endeavoured to construct monetary measures that are comparable across alternative sets of prices. Examples are real GDP and its components; the GDP deflator; the index of the cost of living; cost-benefit analysis. A number of theories about how such measures should be constructed exist; they will be discussed the next section. Here I indicate briefly why I have found them to be unsatisfactory. 1. The different theories have been largely unrelated to each other; if the problem across all applications is essentially the same, that of maintaining a stable money metric, then it is not clear why more than one theory is need. 2. Some of the theories, as in the axiomatic approach, are not based on economics, at least not in a straight forward manner. It is not clear what inferences one can draw from the formulas derived there. 3. Most existing theories have serious defects; either they are based on unrealistic assumptions, or they limit themselves to problems of limited relevance.. For example, many of the economic theories deal with the problem of constructing and index of the cost-of-living for a single consumer with homothetic preferences. But the result is then applied to an aggregate of a large number of diverse consumers with non-homothetic preferences. 4. Remarkably, almost no theory exists on how to define components of a real value, such as the components of real GDP. I argue that the statistical agencies producing national income and product (NIPA) statistics have not found a satisfactory way of doing this.

In this paper I provide a unified theory that has the following features: 1. It is free of unreasonable assumptions such as homotheticity or the existence of a representative agent. 2. It works in different contexts; a single maximizing agent; a group of such agents; more generally when maximizing agents are not explicitly postulated. 3. It demonstrates the existence of a single index formula that can accomplish all of this; it is the Törnqvist index. 4. By demonstrating that the Törnqvist index of applied welfare economics is a quadratic approximation to the Divisia index of theoretical welfare economics a link between these two fields is established.

It is useful to start with some definitions and conceptual clarifications. A basic concept for our purposes is that of *value* defined as a vector product $\mathbf{px}$ of a price and a quantity vector. Usually, we will be interested in values that have featured in a transaction where the value is an *income* to one side, and an *expense* to the other. Depending on the context, I will sometimes use these more specific terms. An unadjusted value is *nominal*; one from the effects of prices changes have, by some means or other been removed is r*eal*. Interest is focused on computing the changes in real values, usually but not always in ratio form, which makes the changes independent of the units of measurement. Measurement formulas expressed as ratios are usually referred to as *indexes* and their theory, rather inappropriately in my view, as *index number theory*; I will instead use the term *index theory*. *Levels* may be defined subsequently by starting with the nominal value of a base year and then extrapolating the computed increments of real value. An example would be GDP at the prices of some base year.

The interpretation of real values has been the subject of a fallacy that has permeated both the construction and use of statistics. It is the idea that directly computed real values are in some sense aggregated quantities and can be treated as though they were quantities. Regardless of how widespread this practice is, no justification for it has, or can be given.[1] The

[1] According to the composite commodity theorem of Hicks, a subgroup of commodities can be treated as though it were a single commodity if the prices of all commodities in this group always move in the same proportion. This highly special case would not require the use of an index.

term ‘quantity index’ is also indicative of this confusion; however, since it is so well established, I continue to use it.

Even though the theories that have been developed to measure real values differ greatly, they all follow the same basic concept: The change in nominal value is decomposed into a change in real value, associated with the changes in quantities and an inflation associated with the changes in prices. This definition may be puzzling at first, since inflation is usually defined as the average rate of increase of prices. But the two definitions are equivalent, as will become particularly clear in relation to the Divisia and Törnqvist indexes. A simple illustration can be given here: If prices double, the nominal expenditure doubles also and the value of the monetary unit (the metric) is cut in half. Because prices and quantities enter values symmetrically, the two measures are also symmetric and usually computed by means of indexes of the same form. I will refer to the relationship between quantity changes and value changes as the *real value metric* and to the relationship between price changes and value changes as the *inflation metric.* When referring to both I will use the term *money metric*. Most of the conditions postulated in axiomatic index theories are properties of the money metric.

Why compute real values at all? I think that the answer is fairly obvious, though in the relevant theories surprisingly little has been said on the subject. Both individual and groups feel (rightly or wrongly) that they are better of if they can have a larger command over resources. In the constant price case this can be measured by the size of their budget, when prices are variable, the same information is conveyed, at least approximately, by the computed real value of their budget. The reticence in explaining the relevance of real values is due to the extremely restrictive ‘welfare’ concept to which economic theorists have largely been committed. The two concepts usually employed in relation to ‘welfare’ are the Pareto optimum and the social welfare function. For the construction of empirical measures these have been largely useless, but they have led to reluctance to refer to real value as a welfare measure. In practice, when trying to form a judgment about how well off a society is, we look at many different statistics; real income is one, but statistics on health, on crime and on other aspects of social life are also important. None of these measures the ‘happiness’ of individuals but they are all relevant for judging the quality of life. I have proposed to refer to such measures as *welfare indicators.*

2. EXISTING THEORIES FOR THE MEASUREMENT OF REAL VALUES AND INFLATION

In the following I give a very brief, critical survey of the existing theories. They are: 1. Axiomatic index theory. This is followed by four economic theories: 2. Consumer surplus. 3. The econometric theory of welfare measurement. 4. The theory of superlative indexes. 5. The theory of Divisia indexes. 5. The question of how to compute the components of a real value, such as the components of real GDP. 6. The final topic is one that has been of great importance not only in regard to measurement, but to macroeconomics generally: the use of representative agent models. References that discuss these theories in depth are given in the next paragraph.

The most prolific contributor to various theories of economic measurement in recent decades has been Erwin Diewert. A comprehensive and up to date survey of topics 1. and 4. is found in Diewert (2007a). Diewert and Nakamura (1993) contains many of Diewert’s original papers as well as historical material. Topic 2. is treated in Diewert (2007b) and in Hillinger (2001). Regarding Topic 3, Slesnick (1998) is a survey; Jorgenson (1990) is the most ambitious implementation of the theory. The theory and history of Divisia indexes is covered in Balk (2005). Surprisingly, hardly any theoretical literature exists regarding 5. A large but scattered literature deals with representative agents. I have surveyed this literature in Hillinger

(2006). A book on the subject that contains much historical information is Hartley (1997). Finally, a precursor of the present paper is Hillinger (2003).

2.1. The axiomatic theory

The axiomatic theory has several weaknesses. **a.** It does not provide an economic theory that would indicate to what problems the proposed measures can provide a solution. **b.** While most proposed axioms have an intuitive plausibility, their origin and precise justification remains unclear. **c.** Those axioms that can be given an economic interpretation are generally not sufficient to derive a specific formula. The criticisms under **a** and **b** are closely related. For most of the axioms the criticism can actually be met by interpreting them as manifestations of the money metric. This will be elaborated below.

To exemplify my argument I refer to two recent contributions to the axiomatic theory both of which lead to the Törnqvist price index that is also central to the present paper. Let $\mathbf{p}^t, \mathbf{x}^t$ be the price and quantity vectors at time *t*, and $v^t = \mathbf{p}^t \mathbf{x}^t$ the corresponding value and $s_i^t = p_i^t x_i^t / \mathbf{p}^t \mathbf{x}^t$ the value share of the *i*th commodity. The Törnqvist (1936) quantity and price indexes are defined by

(2.1) $$Q_T = \prod \left(\frac{x_i^1}{x_i^0} \right)^{\bar{s}_i}, P_T = \prod \left(\frac{p_i^1}{p_i^0} \right)^{\bar{s}_i}, \quad \bar{s}_i^t = \frac{1}{2}\left(\alpha_i^{t-1} + \alpha_i^t \right).$$

Diewert (2004) presents 17 axioms that together imply the Törnqvist price index. Of these, 14 can be interpreted as aspects of the inflation metric in the following sense: Let $P\left(\mathbf{p}^0, \mathbf{p}^1, \mathbf{v}^0, \mathbf{v}^1\right)$, where $\mathbf{v}$ is a vector of values, be the bilateral price index for the indicated two periods. *P* has a property of the inflation metric if

(2.2) $$P\left(\mathbf{p}^0, \mathbf{p}^1, \mathbf{v}^0, \mathbf{v}^1\right) = \frac{\mathbf{p}^1 \mathbf{x}}{\mathbf{p}^0 \mathbf{x}},$$

where $\mathbf{x}$ is some fixed quantity vector and the variation in $\mathbf{p}$ is such that the resulting change of the expression can be deduced from general principles. For example, if all prices change in the same proportion, the index must change in that proportion also. Of the 17 axioms 14 satisfy this condition. Let us look at those that do not. The three conditions are shown below. Above each is the numbering and label given in the original.

T11: *Transitivity in Prices for Fixed Value Weight:*

(2.3) $$P\left(\mathbf{p}^0, \mathbf{p}^1, \mathbf{v}^r, \mathbf{v}^s\right) P\left(\mathbf{p}^1, \mathbf{p}^2, \mathbf{v}^r, \mathbf{v}^s\right) = P\left(\mathbf{p}^0, \mathbf{p}^2, \mathbf{v}^r, \mathbf{v}^s\right).$$

T12: *Quantity Weights Symmetry Test*:

(2.4) $$P\left(\mathbf{p}^0, \mathbf{p}^1, \mathbf{v}^0, \mathbf{v}^1\right) = P\left(\mathbf{p}^0, \mathbf{p}^1, \mathbf{v}^1, \mathbf{v}^0\right).$$

I give a verbal statement for the next axiom, since it is a bit complex to state symbolically:

T16: *Own Share Price Weighting*:

If all prices are fixed except one, then the index depends only on that price and its value share.

It is easy to check that P_T satisfies these axioms. In fact, the only rationale for introducing them appears to be that, along with the other axioms, they enable the deduction of P_T.

One criticism of the axiom system made by Diewert himself can be ameliorated in the present paper. He noted that a symmetric set of axioms can be used to derive Q_T but that the two indexes are not dual in the sense that

(2.5) $$P_T Q_T \neq \frac{v^1}{v^0}.$$

I argue below that the property holds to a quadratic approximation which is good enough for applications.

Another set of axioms for P_t will be discussed in the next section.

Economic Theories of Welfare Measurement:

2.2. Consumer Surplus

Consumer surplus (CS) has had a long and rather confused history and there is neither a unique formula nor a unique terminology associated with it. The usual geometrical derivation derives the benefit to a consumer of the reduction in the price of some good in terms of areas under a demand curve as

$$CS = -\frac{1}{2}\left(x^0 + x^1\right)\left(p^1 - p^0\right). \tag{2.6}$$

If money expenditure remains constant, this is equivalent to

$$CS = \frac{1}{2}\left(p^0 + p^1\right)\left(x^1 - x^0\right), \tag{2.7}$$

a formula often use in project evaluation.

Much of the appeal of CS is due to the fact that the derivation is based on a simple, intuitive and economic argument, yielding a simple expression that can be easily computed from data. Moreover, importantly for applications, the measure is evidently additive, so that the formula applied to aggregate data yields the aggregate CS.

Already Alfred Marshall had put his finger on an essential weakness of the intuitive derivation: The argument implicitly assumes that, as one moves along the demand curve, successive increments expenditure cause equal changes in utility[1]. Much later, Samuelson (1942) proved that this condition 'constancy of the marginal utility of income' cannot possibly hold. There are some well known analytical derivations of CS that are often cited in defense of its use When one analyses these carefully, one finds the implicit assumption of a constant marginal utility of income. Another fundamental difficulty is that when more than one price and or income change, stable demand functions are no longer defined.[2] All of these difficulties have not deterred the advocates of applied cost-benefit analysis. For example Layard and Glaister (1994) write:

> This is a formula which is used over and over again in cost-benefit analysis, especially for small changes in prices so the linearity assumption is a reasonable approximation to *any* actual demand curve. (p.4)

To analyze the general case, when all prices and quantities are variable, define the centered price difference

$$CPD = \frac{1}{2}\left(\mathbf{x}^0 + \mathbf{x}^1\right)\left(\mathbf{p}^1 - \mathbf{p}^0\right) \tag{2.8}$$

and the centered quantity difference

$$CQD = \frac{1}{2}\left(\mathbf{p}^0 + \mathbf{p}^1\right)\left(\mathbf{x}^1 - \mathbf{x}^0\right). \tag{2.9}$$

The two differences decompose a change in value:

$$v^1 - v^0 = CPD + CQD. \tag{2.10}$$

Diewert (1992) focused on *CQD* as a measure of a consumer's welfare change and obtained various approximation results. In Hillinger (2001) I treat *CPD* and *CQD* jointly as measures of a consumer's theoretical cost-of-living and real consumption, focusing on the non-homothetic case. Using symmetrical definitions of the theoretical measures, I was able to validate and extend the quadratic approximation result of Hicks.

In spite of these positive results, I became disillusioned with welfare measures expressed as differences. The principal difficulty is that they are not invariant to the choice of units of measurement. This is both inconvenient and at time leads to pathological results. Thus Diewert (1976b) has shown that in some situation a proportional increase of prices and

[1] Marshall's views on this issue are discussed in some detail by McKenzie (1983).

[2] For a fuller discussion of these issues and references to the relevant literature see Hillinger (2001).

expenditures, leaving the quantities of goods unchanged, may change the sign of *CQD*. This problem can be ameliorated by deflating $\mathbf{p}^1$ back to the level of $\mathbf{p}^0$, but this introduces another and rather inelegant complication. I turned away from these measures and developed instead the theory of the present paper.

Two Modern Theories

The two theories to be discussed under this heading have been the subject of intensive efforts on the part of mathematical economists and econometricians over the past several decades. While there are important differences between them, there is also a substantial common ground in the form of the assumption of a 'flexible functional form' of a quadratic in the logarithms of the inputs to the aggregator function; usually the utility function of a consumer, but equally the production function of a producer. This function gives a quadratic approximation to an arbitrary well behaved aggregator function.

2.3. The Econometric Approach to Welfare Measurement

At the center of this approach is a methodology that is referred to as 'exact aggregation'. It imposes very strong and in my view implausible conditions: The utility functions are homothetic and identical except for a vector of demographic characteristics. The method is applied without testing the validity of these assumptions. Furthermore, highly aggregated quantity indexes instead of actual commodities are used in the estimates. No justification for doing this is given. Finally, the approach attempts to go beyond the determination of real values to the determination of a distributionally sensitive social welfare function. However, no generally accepted social welfare function exists, and the one employed in this context has a parameter that has to be fixed quite arbitrarily. Quite independent of these criticisms is the fact that the very complexity of the approach has prevented its adoption by statistical agencies.

2.4. The Theory of Superlative Indexes

The theory of superlative indexes has the same starting point as the econometric theory, namely the assumption of a flexible functional form. From that initial position, the two theories go off in different directions. The econometric theory assumes that the flexible functional form can be estimated directly on the assumption that it can be used with a few quantity indexes representing broad categories of goods. The theory of superlative indexes makes no such assumption and stays at the level of individual commodities. The basic result is that a family of 'superlative' indexes reproduce the changes measured by the flexible functional form.

As in the econometric theory, homotheticity is the usual assumption in this theory also. However, the theory was also applied to the non-homothetic case, when the Törnqvist index emerges as the relevant superlative index. This part of the theory is closely related to the theory of the present paper and will be discussed further in Section 4.2…The superlative theory does not extend directly to groups, however the results on the aggregation of Törnqvist indexes given in Section 5 could be used to remedy this shortcoming.

2.5. Divisia and Törnqvist Indexes

So far we have not found a theory for the measurement of real value and inflation that is completely satisfactory in the sense of being rigorous, based on plausible assumptions and applicable to all the situations in which such indexes are used. In the natural sciences this kind of problem is usually simplified by taking limits, thus analyzing the situation at a point. The interval is dealt with subsequently by using integrals or differential equations. Similar approaches were suggested by Bennett (1920) and Divisia (1926). Bennet noticed that

(2.11) $$dv = \mathbf{x}d\mathbf{p} + \mathbf{p}d\mathbf{x}$$

and interpreted the differentials as being those of price and quantity indexes:

(2.12) $$dP_B = \mathbf{x}d\mathbf{p}, \quad dQ_B = \mathbf{p}d\mathbf{x}\,.$$

Divisia realized that it is better to deal with proportional changes that are invariant to the units of measurement and transformed the Bennett differentials accordingly. Divisia differentials and integrals are treated in detail in the following section. Here I mention only the two fundamental problems connected with this approach, that have thus far not had a satisfactory resolution. The first is the question of how to approximate the Divisia integrals over an interval. Törnqvist had noticed that when expenditure shares are constant; the integral corresponds exactly to the index now known by his name, the shares taking on the common value. For the non-constant case, Törnqvist proposed the use of the average shares, but provided no formal justification. An even more serious problem is that the partial differentials that define the indexes are path dependent. I believe that the present paper is the first to provide a convincing solution to these problems.

2.6. Real Value and its Components

There has been virtually no theorizing on how the components of a real aggregate should be determined. Given the importance of the components of real GDP, this lack of interest on the part of theorists is hard to understand. The most elementary notion that one can have about the parts of a total is that they should add up. National income statisticians, have strongly felt this intuition, but they have found it difficult, if not impossible, to implement. For many decades after the establishment of the accounts, the practice has been to report all real magnitudes at constant base period prices, which maintains additivity. The problem is that as the base year recedes, these prices become more and more irrelevant in relation to current transactions. The need then arises to choose a new up-to-date base and to convert the old data to the new base so as to obtain consistent time series over the entire time span. For this purpose a scale factor has to be used such that for the year of the transition; the old data are scaled to the levels of the new. The problem is that if the scale factor that is relevant for the aggregate is used for the components also, these show large discontinuities that do not correspond to the actual evolution of the sectors. Alternatively, if the sectors are scaled individually, additivity is lost. In practice the latter method was usually employed and additivity restored by simply redistributing the discrepancy over the sectors. These arbitrary manipulations reduce the sophisticated econometric methods that employed the data to absurdity.

Still another methodology is of more recent origin and was adopted mainly by English speaking countries. Here a symmetric, quantity index, usually of the Fisher type, is used in chained form to compute independently each component and the total as real values. The components do not add to the total and the discrepancy is published. A structured macroeconometric model cannot be estimated from these data. The most reasonable assumption that can be made about this discrepancy is that it will behave as a random walk, without any tendency to return to a zero mean; it will tend to grows with time, so that some further arbitrary adjustment will eventually be required.

Having essentially completed the present paper, I obtained a copy of Lequiller and Blades (2006), the most recent comprehensive OECD publication on the national accounts. In Chapter 2 they discuss the procedures used to create real (in their language 'volume') accounts. They are quite critical regarding non-additive sectoral accounts computed by means of chained quantity indexes and state that this practice is followed only by the US and Canada. The methods used by other countries and by OECD itself are described somewhat sketchily. My understanding of their account is that real sector levels are obtained by extrapolating a base year using a chained Laspeyres quantity index. Real GDP is then defined as the sum of the sectors. This procedure has the consequence that the share of the real sectors in the total will drift away from those of the nominal shares. This in turn means that the implied *relative prices* between the real sectors are not the actual relative prices at which market transactions can take place. A model based on such data cannot be an adequate representation of an economy.

2.7. The Representative Agent

Regardless of how an index is related to the concept of a maximizing agent, when it is applied to aggregate data, the justification usually involves a reference to a representative agent. For example, in relation to the cost-of-living index (COLI). Schultze and Mackie (2002) state:

> The concept of the "representative consumer" frequently comes up in discussions of COLIs and of price indexes more generally. Indeed, it is often difficult to discuss COLIs with non-economists, policy makers, or the public at large without some sort of appeal to the concept. Sometimes the use is ambiguous or implicit: For example, a COLI might be presented in terms of the amount of money needed to keep consumers, or even "the consumer" as well off as before the price change. Or it might appear in thinking about the change in expenditure that would be necessary to offset the effects of inflation on "consumer living standards." Similar phrases are often used to describe substitution effects in response to price changes. Sometimes the language refers explicitly to the representative consumer, sometimes to a "typical" or "average" consumer. (p. 241-2).

While the use of the concept described here is informal, the concept is also dominant in formal modeling in contemporary macroeconomics and welfare economics and in econometric work done in these fields. This in spite of a substantial literature that has shown that the concept cannot be justified on the basis of realistic assumptions. Here I will quote from a contribution regarding the representative consumer:

> Given the arguments presented here – that well-behaved individuals need not produce a well-behaved representative agent; that the reaction of a representative agent to change need not reflect how the individuals of the economy would respond to change; that the preferences of a representative agent over choices may be diametrically opposed to those of society as a whole – it is clear that the representative agent should have no future. (Kirman, 1992, p. 134).

A final quotation is from Deaton and Muellbauer (1980):

> These aggregation conditions often turn out to be stringent, which has tempted many economists to sweep the whole problem under the carpet or to dismiss it as of no importance. (p. 148).

The literature on representative agents deals only with the aggregation of individual commodities over agents, but that does not describe the situation when these models are used empirically. In applications real expenditure indexes, aggregated over many thousands of diverse commodities, are treated as though they were the individual commodities of economic theory. No justification for this is ever given. The representative agent as a concept for the use of aggregate date is simply invalid. A central message of this paper is that the exact aggregation and interpretation of indexes is possible without it.

3. DIVISIA INTEGRALS AND TÖRNQVIST INDEXES

3.1. Bennet and Divisia Differentials

The Bennett differentials

(3.1) $$dv = QdP + PdQ, \quad QdP = \mathbf{x}d\mathbf{p}, \quad PdQ = \mathbf{p}d\mathbf{x}$$

provide the starting point of the analysis.

Divisia converted the Bennet differentials to proportional form, which makes them independent of units of measurement. The Divisia price differential is

(3.2) $$\frac{PQ\dfrac{dP}{P}}{PQ} = d\ln P = \frac{\sum p_i x_i \dfrac{dp_i}{p_i}}{y} = \sum s_i \, d\ln p_i \ .$$

Similarly, the Divisia quantity differential is

(3.3) $$d\ln Q = \sum s_i \, d\ln x_i \ .$$

The two differentials decompose the change in value:

(3.4) $$d\ln v = d\ln P + d\ln Q.$$

The decomposition has two paramount features:

a. The real growth rate of the value is a weighted average of the quantity growth rates and the inflation rate is a weighted average of the proportional price changes, the weights being the average expenditures shares. This illustrates the statement made earlier that inflation can be interpreted as either the average growth rate of prices or as the growth rate of value caused by the price changes. **b.** Real growth and inflation rates are dual so that real growth computed directly, or indirectly via deflation, has the same value.

3.2. Divisia Integrals

The point decomposition (3.4) can only be a starting point, since in an empirical context we will always be interested in comparing two or more distinct observations. A step in that direction is to define the integrals corresponding to the Divisia differentials.

The Divisia price and quantity integral are

$$I_P = \ln\frac{P^1}{P^0} = \int_0^1 \sum s_i(\tau)\frac{p_i'(\tau)}{p_i(\tau)}d\tau, \quad p_i'(\tau) = \frac{\delta p_i}{\delta\tau}, \tag{3.5}$$

$$I_Q = \ln\frac{Q^1}{Q^0} = \int_0^1 \sum s_i(\tau)\frac{x_i'(\tau)}{x_i(\tau)}d\tau, \quad x_i'(\tau) = \frac{\delta x_i}{\delta\tau}. \tag{3.6}$$

These integrals are path-dependent. Their sum is the integral of the total differential of logarithmic expenditure and thus path-independent:

$$I_P + I_Q = \int_0^1 d\ln v(\tau) = \ln\frac{v^1}{v^0}. \tag{3.7}$$

The Divisia indexes corresponding to the integrals are

$$P_D = \frac{P^1}{P^0} = \exp I_P, \quad Q_D = \frac{Q^1}{Q^0} = \exp I_Q. \tag{3.8}$$

3.3. Quadratic Approximation

Before proceeding to a formal analysis, I will give here a verbal discussion of how I propose to deal with the conceptual problems that have bedeviled the analysis of Divisia integrals. I use a combination of economic and mathematical arguments. The economic argument is that the values of the derived price and quantity indexes should depend solely on prices and quantities at the end points of the interval. This is the standard assumption that has always been made in index theory. It should be noted that an influence of the path on the outcome is by no means excluded. The assumption is only that whatever outcome is reached, the price/quantity data of the initial and final situations are all that is needed for a comparison.[1] The mathematical result is that the Divisia integral is approximated quadratically if prices and quantities grow exponentially or more generally monotonically, over the interval being considered. These arguments together provide a strong, though not the only justification for accepting the Törnqvist index.

I give two slightly different proofs of the quadratic approximation property of Törnqvist indexes. The first assumes that all variables grow at constant rates. This is the most reasonable assumption one can make if one assumes a specific path. This path can also be given a normative interpretation: If the actual path is unknown, than the integral should be given the value associated with the most regular path. The second proof only requires the assumption of monotone paths. Both proofs are based on the

Trapezoid Rule: [2]

$$\int_a^b f(x)dx = \frac{b-a}{2}\left[f(a)+f(b)\right] - \frac{(b-a)^3}{12}f''(c), \quad c\in[a,b]. \tag{3.9}$$

[1] Some writers have argued that the path does contain additional information (See Balk 2005), however, neither has the nature of that information ever been made clear, nor has anyone shown how to extract it.

[2] For a discussion of the rule and related results see Judd (1998, Section 7.1). The trapezoid rule is closely related to the quadratic approximation lemma given below.

The first term on the right is the trapezoidal approximation to the area above (or below) the interval *b-a*, based on the height of the function at the endpoints. The second term is the residual, which is cubic in Δx.

The theorem will be proven in relation to the price index, the case of the quantity index being analogous. In order to employ the scalar form of the trapezoid rule we write the ith component as

(3.10) $$I_{iP} = \int_0^1 s_i\left(\tau\right)\frac{p_i'\left(\tau\right)}{p_i\left(\tau\right)}d\tau.$$

First Törnqvist Approximation Theorem:

Assume that prices and quantities grow at constant rates. Then

(3.11) $$\exp I_p = \frac{P^1}{P^0} = P_T + O_3, \quad \exp I_Q = \frac{Q^1}{Q^0} = Q_T + O_3.$$

Proof:

Letting r_i be the rate for the ith price, it is determined by

(3.12) $$p_i^1 = p_i^0 \exp r_i, \quad \Rightarrow r_i = \ln\frac{p_i^1}{p_i^0}.$$

Then

(3.13) $$I_P = \int_0^1 \sum s_i(\tau)\ln\frac{p_i^1}{p_i^0}d\tau.$$

The ith component

(3.14) $$I_{iP} = \int_0^1 s_i(\tau)\ln\frac{p_i^1}{p_i^0}d\tau$$

is of the standard form given in (3.9), so that

(3.15) $$I_{iP} = \overline{s}_i \ln\frac{p_i^1}{p_i^0} + O_3(\Delta\tau), \quad \overline{s}_i = \frac{1}{2}(s_i^0 + s_i^1).$$

It follows that to a quadratic approximation

(3.16) $$I_P = \sum V_{iP} = \sum \overline{s}_i \ln\frac{p_i^1}{p_i^0} = \sum \ln\left(\frac{p_i^1}{p_i^0}\right)^{\overline{s}_i}$$

and

(3.17) $$\exp I_P = \prod\left(\frac{p_i^1}{p_i^0}\right)^{\overline{s}_i} = P_T.$$

Putting the results together, we can write the approximation of the theoretical Divisia price index by the Törnqvist price index as

(3.18) $$P_D = \exp I_P = P_T + O_3(\Delta\tau).$$

An analogous derivation for the Törnqvist quantity index gives

(3.19) $$Q_D = \exp I_Q = Q_T + O_3(\Delta\tau).$$

The assumption of constant growth rates is the most natural and simplest assumption that can be made in order to prove the quadratic approximation property of Törnqvist indexes. Nevertheless, it is interesting to ask if the result holds under more general conditions. This is the subject of the

Second Törnqvist Approximation Theorem: Assume that: prices and quantities grow monotonically in the interval (0, 1)*. Then* (3.18) *and* (3.19) *hold.*
Proof:
Given the monotonicity assumption, each value of p_i in the interval (0, 1) is unique. Symbolically we can represent the value share at p_i and hence also at $\ln p_i$ as a function $s_{i01}\left(\ln p_i\right)$. This function has no causal significance; it can in principle be constructed ex post after a given monotone realization over the interval. It will vary from interval to interval, hence the subscript.

(3.20) $$I_{iP} = \int_{\ln p_i^0}^{\ln p_i^1} s_{i01}\left(\ln p_i\right) d\ln p_i \ .$$

This expression is of the form given in(3.9) so that

(3.21) $$I_{iP} = \overline{s}_i (\ln p_i^1 - \ln p_i^0) + O_3(\Delta \ln p_i),$$

which is analogous to (3.15). The implication is that (3.18) and (3.19) hold.

Third Derivation of the Törnqvist Index

While working on previous drafts of the present paper I had the recurrent thought that a more direct derivation of the Törnqvist index, as the embodiment of the money metric should be possible. Several attempts in this direction failed until I hit on what now seems to me to be the simplest and most direct formulation. The starting point is again provided by the Bennett differentials

(3.22) $$dv = QdP + PdQ, \quad QdP = \mathbf{x}d\mathbf{p}, \quad PdQ = \mathbf{p}d\mathbf{x} \ .$$

Divisia and those who have followed in his footsteps have implicitly regarded the differentials in (3.22) as partial differentials of functions of the 2*N* prices and quantities. There is however another interpretation that turns out to be more tractable. The alternative is to define the functions $Q\left(\mathbf{x}\right), P\left(\mathbf{p}\right)$ with the understanding that prices act as time varying parameters in $Q\left(\mathbf{x}\right)$ and quantities similarly in $P\left(\mathbf{p}\right)$. Together they decompose the nominal expenditure

(3.23) $$Q\left(\mathbf{x}\right)P\left(\mathbf{p}\right) = v = \mathbf{px} \ .$$

The Divisia price differential is

(3.24) $$\frac{PQ\dfrac{dP}{P}}{PQ} = d\ln P = \frac{\sum p_i x_i \dfrac{dp_i}{p_i}}{y} = \sum s_i \, d\ln p_i\,, \quad s_i = \frac{p_i x_i}{y}.$$

Similarly, the Divisia quantity differential is

(3.25) $$d\ln Q = \sum s_i \, d\ln x_i \ .$$

Since $Q(\ )$ and $P(\ )$ are now functions, we can interpret the value share s_i appearing in (3.25) and (3.24) as the slopes of these functions. The changes of the functions over an interval can then be computed directly and to a quadratic approximation by using the *Quadratic Approximation Lemma*:[1]

Given the quadratic function $f\left(\mathbf{z}\right) = a + \mathbf{az} + \frac{1}{2}\mathbf{zAz}$

[1]The lemma is discussed in Diewert (1976a) and used there for a different derivation of the Törnqvist index in the context of the economic theory of indexes. For an exhaustive treatment of the lemma and its applications in index theory see Diewert (2000).

(3.26) $$f\left(\mathbf{z}^1\right)-f\left(\mathbf{z}^0\right)=\frac{1}{2}\left[\nabla f\left(\mathbf{z}^0\right)+\nabla f\left(\mathbf{z}^1\right)\right]\left(\mathbf{z}^1-\mathbf{z}^0\right).$$

Applying the lemma gives

(3.27) $$\ln Q_1-\ln Q_0=\sum\frac{1}{2}\left(s_{i0}+s_{i1}\right)\left(x_{i1}-x_{i0}\right),$$

the defining equation for the Törnqvist quantity index. The derivation of the Törnqvist price index is analogous.

This derivation of the Törnqvist indexes is simpler, more straight forward and stronger than the derivations based on approximations of integrals. The quadratic approximation property has now been shown to hold regardless of the path. Assuming a continuous path between the endpoints, the new interpretation does not do away with path dependency. Even with given endpoints, differences in the slope parameters along the path would cause different changes of the values of the functions. The quadratic approximation property to the path is not affected by path dependency.

3.4. Axiomatic Derivation and Interpretation of Törnqvist Indexes

In this section I discuss A particularly concise and elegant derivation of the Törnqvist price index that is due to Balk and Diewert (2001). Their derivation is based on three assumptions:

The Index is a Function of Value Shares and Price Ratios:

(3.28) $$\ln P\left(\mathbf{p}^0,\mathbf{p}^1,\mathbf{s}^0,\mathbf{s}^1\right)=\sum m_i\left(s_i^0,s_i^1\right)\ln\left(\frac{p_i^1}{p_i^0}\right),$$

where $m_i\left(\ \right)$ is an, as yet unspecified, averaging function. The authors further assume two of the most basic axioms of the inflation metric:

Proportionality in Current Prices:

(3.29) $$P\left(\mathbf{p}^0,\lambda\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1\right)=\lambda P\left(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1\right),\text{ for }\lambda\succ 0.$$

Inverse Proportionality in Base Period Prices:

(3.30) $$P\left(\lambda\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1\right)=\lambda^{-1}P\left(\mathbf{p}^0,\mathbf{p}^1,\mathbf{x}^0,\mathbf{x}^1\right),\text{ for }\lambda\succ 0.$$

Balk and Diewert show that (3.28), (3.29) and (3.30) imply

(3.31) $$P=P_T.$$

The authors considered only the derivation of a price index. The Törnqvist quantity index could be derived from analogous axioms applied to the quantities.

Regarding any set of axioms we should ask where they come from. Axioms (3.29) and (3.30) reflect fundamental properties of inflation. Axiom (3.28) is most naturally interpreted as extending the properties of the Divisia differentia (3.2) to an interval. The instantaneous change $d\ln p_i$ is replaced by its integral over the interval, $\ln\left(\frac{p_i^1}{p_i^0}\right)=\ln p_i^1-\ln p_i^0$; the instantaneous share α_i is replaced by an (as yet unspecified) average $m_i\left(s_i^0,s_i^1\right)$. The axioms together imply that the only functional form that is compatible with this choice of variables and properties of inflation is P_T. In my interpretation, the Balk/Diewert axioms provide an alternative derivation of the Törnqvist indexes from the Divisia differential.

The derivation of the Törnqvist Index from the Divisia integral has a further advantage over the pure axiomatic derivation. In the context of the usual axiomatic approach, it is regarded as a defect of the Törnqvist indexes that they do not satisfy the duality $v^1/v^0=QP$. Since the duality is satisfied by the Divisia indexes, it is satisfied by the Törnqvist indexes to a quadratic approximation. For practical purposes one can equally well compute real value growth directly with Q_T, or indirectly by deflating with P_T.

4. THE RATIONALITY ASSUMPTION

4.1. The Continuous Approach

The existing literature on the application of the Divisia index to the problem of the utility maximizing consumer has focused on the assumption of homotheticity. This leads to an elegant theory that avoids the path dependency of the usual Divisia index.[1] In this section, I present the Divisia theory for the non-homothetic, but rational consumer (household)[2].

The following definitions will be used: Let $\mathbf{x}$ be the household consumption vector, $\mathbf{p}$ the corresponding price vector, $y = \mathbf{px}$ the household expenditure and $u(\mathbf{x})$ a utility function, assumed twice continuously differentiable and strictly quasi-concave. The corresponding expenditure function

(4.1) $$e(\mathbf{p},u) = \min_{\mathbf{x}} \ \mathbf{px}\colon u(\mathbf{x}) \geq u$$

specifies the minimum expenditure required to reach the utility level u at prices $\mathbf{p}$. The expenditure function is the fundamental tool for aggregating prices and quantities in this context. How this is to be done in the general non-homothetic case has not been clarified in the received theory. I propose to do this analogously to the preceding sections by using continuity in order to arrive at unambiguous parameterizations. I also adopt a terminology appropriate for the consumer sector: the inflation measure will now be referred to as the *cost-of-living* (C) and the real expenditure measure as *real consumption* (R). We now require that

(4.2) $$C(t)R(t) = e(\mathbf{p}(t),u(t)) = y(t).$$

The increment in expenditure due to an increment in the cost-of living is defined as

(4.3) $$RdC = \nabla_{\mathbf{p}} \mathrm{e}(\mathbf{p},u)\,\mathrm{d}\mathbf{p}$$

and the increment in expenditure due to the increment of real consumption by

(4.4) $$CdR = \nabla_{\mathbf{x}} e(\mathbf{p},u(\mathbf{x}))\,d\mathbf{x}.$$

These increments decompose the expenditure change so that

(4.5) $$de = dy = RdC + CdR..$$

These results are analogous to those for the Divisia differentials. The difference is that $\mathbf{x}$ is now not arbitrary, but rather the solution to the consumer's maximization problem (4.1). The money metric is now defined in relation to the differential (4.4) and can be alternatively referred to as money metric utility or real consumption.

Further progress requires the following

Lemmas on duality of the expenditure function

Let $\mathbf{h}(\mathbf{p},u)$ be the Hicksian (compensated) demand function.[3]

(4.6) (Hotelling) $$\nabla_{\mathbf{p}} e(\mathbf{p},u) = \mathbf{h}(\mathbf{p},u) = \mathbf{x}.$$

(4.7) (Balk) $$\nabla_{\mathbf{x}} e\big(\mathbf{p},u(\mathbf{x})\big) = \mathbf{p}\,.$$

Where $\mathbf{x}$ must be the solution to (4.1)

Converting(4.3) to logarithmic form and using (4.6) gives

(4.8) $$\frac{RC\dfrac{dC}{C}}{RC} = d\ln C = \frac{\sum p_i \dfrac{\partial}{p_i} e(\mathbf{p},u)\dfrac{dp_i}{p_i}}{e(\mathbf{p},u)} = \frac{\sum p_i x_i \dfrac{dp_i}{p_i}}{y} = \sum s_i d\ln p_i\,.$$

Similarly, using (4.4) and (4.7)

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[1] This theory is reviewed in Balk (2000, Section 8) and in Diewert (2001, Section D.1)

[2] The conditions under which a household, as opposed to an individual consumer, can be assumed to be utility maximizing are the subject of a literature that began with Samuelson (1956) and was elaborated further by Pollak (1980).

[3] Hotelling's lemma is standard fare of microeconomic textbooks. For the proof of Balk's lemma see Balk (1989, p. 166).

(4.9) $$\frac{CR\,\frac{dR}{R}}{CR} = d\ln R = \frac{\sum x_i \frac{\partial}{x_i} e(\mathbf{p},u)\frac{dx_i}{x_i}}{e(\mathbf{p},u)} = \frac{\sum x_i p_i \frac{dx_i}{x_i}}{y} = \sum s_i d\ln x_i .$$

The logarithmic differentials of *C* and *R* are precisely those obtained earlier in the case of the Divisia inflation and real expenditure differentials. We can therefore use any of the previous approximation results to arrive at the Törnqvist indexes:

(4.10) $$\frac{C^1}{C^0} = P_T + O_3, \quad \frac{R^1}{R^2} = Q_T + O_3 .$$

The interpretation of the Törnqvist index Q_T is now that the proportional increase in money metric utility of the consumer is the same as would have obtained if nominal expenditure had increased in that proportion at constant initial prices. The same interpretation obtains for the indirect measure $\frac{y_1/P_T}{y_0}$.

The interpretation just given is subject to some qualification. *Money metric utility* is defined by the expenditure function $e(\mathbf{p}^0, u(\mathbf{x}))$, for a given base period price vector $\mathbf{p}^0$. From (4.9) it is seen that the Divisia differential for real consumption gives the change in expenditure due to the change in consumption and hence utility *at the instantaneous price*. Integration takes place over a changing money metric. The construction of the Divisia and Törnqvist indexers is such that that they are not affected by scale effects and therefore not by changes in the price *level*, but they can be affected by changes of *relative* prices as well as by the utility level. The results of the next section clarify this matter further.

4.2. The Discrete Approach

The fixation of index theory on the assumption of a homogeneous aggregator function is the more surprising as Theil (1967, 1968), in a brilliant but neglected contribution, developed the theory of the general case for the individual utility maximizing consumer. Only his assumptions and results are given here, the reader is referred to the original paper for the proofs.

Theil begins his analysis by defining the theoretical index of the cost-of-living, also known as the Konüs cost-of-living index.

(4.11) $$P_K\left(\mathbf{p}^1, \mathbf{p}^0; u^*\right) = \frac{e\left(\mathbf{p}^1, u^*\right)}{e\left(\mathbf{p}^0, u^*\right)},$$

where the reference utility level u^* remains to be determined.

The real consumption index, also known as the Allen quantity index, is defined as

(4.12) $$Q_A\left(u^1, u^0; \mathbf{p}^*\right) = \frac{e\left(u^1, \mathbf{p}^*\right)}{e\left(u^0, \mathbf{p}^*\right)},$$

with the reference price vector $\mathbf{p}^*$ to be determined. Theil's definition of real consumption is a version of money metric utility normalized by $\mathbf{p}^*$. He explicitly points out the consequence of non-homotheticity: C is not independent of u^* and R is not independent of $\mathbf{p}^*$. In order to determine $\mathbf{p}^*, u^*$ he assumes that $\mathbf{p}^*$ is an average of $\mathbf{p}^0, \mathbf{p}^1$ and that u^* is determined by the indirect utility function $u^* = u\left(y^*, \mathbf{p}^*\right)$, where y^* is the same average of y^0, y^1, as $\mathbf{p}^*$ is of $\mathbf{p}^0, \mathbf{p}^1$. There follow five elementary conditions of symmetry and homogeneity for the averaging function that narrow it down to the geometric one. Specifically, we must have $p_i^* = \left(p_i^0 p_i^1\right)^{\frac{1}{2}}, \quad y_i^* = \left(y_i^0 y_i^1\right)^{\frac{1}{2}}$.

Having obtained unique expressions for the theoretical indexes, Theil turns to their approximation. I state here only the results:

(4.13) $$P_K = P_T + O_3, \quad Q_A = Q_T + O_3.$$

The theoretical indexes are approximated quadratically by the corresponding Törnqvist indexes.

Diewert (1976a, Theorem 2.16) obtained a similar result for the Törnqvist price index via a different route. He showed that on the assumption that the consumer maximizes a general, quadratic, non-homothetic, translog utility function

(4.14) $$P_K\left(\mathbf{p}^1, \mathbf{p}^0; u^*\right) \equiv P_T, \quad u^* = \left(u^1 u^0\right)^{\frac{1}{2}}.$$

The results of this section can be summed up as follows: The change in household expenditure can be decomposed into two parts. One is the change in real consumption, the other the change in the cost of living. The theoretical magnitudes can be defined by means of continuously changing parameters, or by means of discrete parameters that are averages of values taken at the endpoints. In either case, quadratic approximations are given by the appropriate Törnqvist indexes. It should be mentioned that the continuous theory described in this paper is analytically simpler.

4.3. Homotheticity

The assumption of homothetic preferences has been prominent in theories of welfare measurement and economic theories of index numbers. In the econometric approach to welfare measurement homotheticity enables aggregation over consumers. In the theory of bilateral indexes homotheticity is required in order to obtain 'invariant' indexes, which will be defined below. Finally, in the theory of Divisia indexes homotheticity is the condition for path independence. The role of homotheticity in relation to the last two topics is fully explored in Samuelson and Swamy (1974) and Balk (2005). Here I only report the principal results.

Consider again the theoretical indexes

(4.15) $$P_K\left(\mathbf{p}^1, \mathbf{p}^0; u^*\right) = \frac{e\left(\mathbf{p}^1, u^*\right)}{e\left(\mathbf{p}^0, u^*\right)} \quad \textit{and} \quad Q_A\left(u^1, u^0; \mathbf{p}^*\right) = \frac{e\left(u^1, \mathbf{p}^*\right)}{e\left(u^0, \mathbf{p}^*\right)}.$$

Invariance means that P_K must be independent of u^* and Q_A must be independent of $\mathbf{p}^*$. Writing Q_A as a function of the optimizing consumption vectors and imposing the duality condition gives

(4.16) $$\frac{y^1}{y^2} = P_K\left(\mathbf{p}^1, \mathbf{p}^0\right) Q_A\left(u^1, u^0\right).$$

Samuelson and Swamy prove that (4.16) implies and is implied by the assumption of homothetic preferences.

Samuelson and Swamy also demonstrate that homotheticity is both necessary and sufficient for path independence of the Divisia integrals. Balk proves that under this assumption $P_D = P_K$. From the duality property of these indexes it also follows that $Q_D = Q_A$. The final result is that if the consumer's utility function is Cobb-Douglas, then expenditure shares are constant and the Törnqvist indexes are exact for the Divisia indexes. This was Törnqvist's original insight.

The received theory evidently considers the properties of invariance and path independence as being of great importance, but it is neither made clear whence this importance derives, nor what should be done, given that homotheticity is not a realistic assumption. In the introduction to their paper, Samuelson and Swamy state that only the homothetic case allows *all* of Irving Fisher's conditions to be met; they do not elaborate which conditions are otherwise violated, or why this should be a concern. Following is the final section of their paper, titled *Concluding Warning*:

> Empirical experience is abundant that the Santa Claus hypothesis of homotheticity in tastes and in technical change is quite unrealistic. Therefore, we must not be bemused by the undoubted elegances and richness of the homothetic theory. Nor should we shoot the honest theorist who points out to us the unavoidable truth that in nonhomothetic cases of realistic life, one must not expect to be able to make the naive measurements that untutored common sense always longs for; we must accept the sad facts of life, and be grateful for the more complicated procedures economic theory devises.

The authors do derive some rather intricate bounds on the theoretical indexes for the non-homothetic case. These apply only to the single utility maximizing consumer and have had no consequence for applications.

Regarding Divisia integrals and the issue of path independence, Balk (2005) summarized the existing literature as follow:

> A fundamental property of the Divisia indices, which they share with chain indices, is their so-called path-dependency. Over the years this prop1erty has led to conflicting views among economists. On the one hand, by a minority this intriguing, almost "magical", property was considered as a virtue. It was thought that the Divisia indices somehow track economic reality better than (simple) bilateral indices. On the other hand, quite a number of economists have wrestled with this property as a problem and sought after conditions under which the indices may exhibit path-independency.

This is a description of the very unsatisfactory state of the literature. Apart from the fact that there is no agreement, neither of the two positions offers a solution. The first statement is completely vague; it neither clarifies the nature of the allegedly present additional information, nor offers a means of using it. The second position led to the mathematical results on homotheticity that are not satisfied in the real world.

My view on path-dependence is implicit in my derivation of the Törnqvist index. There I showed that it is not a problem as long as the paths are monotone, since to a quadratic approximation all such paths are valued by the Törnqvist indexes and oscillating paths can have no relevance for the comparison between the endpoints.

5. AGGREGATION OF DIVISIA INDEXES OVER AGENTS AND SECTORS

Up to this point we considered Divisia and Törnqvist indexes as aggregators of prices and quantities pertaining to a single unit, be it a household or a market. This section considers aggregation over such units. Unless dealing specifically with aggregation over households, I will use the term 'sector'. The method of aggregation is essentially the same, only that there are now three different kinds of expenditure shares to be considered: The share of the *i*th good in the *k*th sector s_{ik}, the share of the *i*th good in the total s_i, the share of the *k*th sector's expenditure in the total σ_k. These are related by

(5.1) $$s_i = \sum_k s_{ik}\sigma_k, \quad i \in \left(1,\cdots,I\right), \quad k \in \left(1,\cdots,K\right).$$

The logarithmic Divisia price index for the aggregate is

(5.2) $$\begin{aligned} I_P &= \ln\frac{P^1}{P^0} = \int_0^1 \sum_i s_i\left(\tau\right)\frac{p_i'\left(\tau\right)}{p_i\left(\tau\right)}d\tau \\ &= \int_0^1 \sum_k \sigma_k\left(\tau\right)\sum_i s_{ik}\left(\tau\right)\frac{p_i'\left(\tau\right)}{p_i\left(\tau\right)}d\tau \\ &= \sum_k \int_0^1 \sigma_k\left(\tau\right)\sum_i s_{ik}\left(\tau\right)\frac{p_i'\left(\tau\right)}{p_i\left(\tau\right)}d\tau. \end{aligned}$$

Denoting by P_k the Divisia price index for the kth sector, we can also write

(5.3) $$I_P = \sum_k \int_0^1 \sigma_k(\tau)\, d\ln P_k(\tau).$$

Similarly,

(5.4) $$I_Q = \sum_k \int_0^1 \sigma_k(\tau)\, d\ln Q_k(\tau).$$

The aggregate integral is a weighted average of the instantaneous Divisia differentials, the weights being the instantaneous market shares. This is analogous to how the price or quantity changes are weighted in the single sector Divisia differential.

The aggregation properties of the Divisia index are all that is really needed since they are inherited by the Törnqvist index. Nevertheless, it is interesting to directly derive the corresponding results under the rationality assumption of the preceding section. Also interesting is the direct derivation of the aggregation properties of the Törnqvist index. These are the subjects of the next two sections.

6. DIVISIA AGGREGATION OVER RATIONAL HOUSEHOLDS

The theory for the individual household can be extended to an aggregate of households on the assumption that the market price is the same for all consumers. The kth consumer, $1 \le k \le K$, has expenditure y_k and faces market prices $\mathbf{p}$. The aggregate consumption vector is $\mathbf{x} = \sum \mathbf{x}_k$. The collection of utilities is $\mathbf{u} = (u_1, ..., u_K)$. Aggregate expenditure is $y = \Sigma y_k = \mathbf{p}\Sigma\mathbf{x}_k = \mathbf{px}$. Define $\mathbf{X} = (\mathbf{x}_1 ... \mathbf{x}_K)$ and the aggregate expenditure function $e(\mathbf{p}, \mathbf{u}(\mathbf{X})) = \sum e_k(\mathbf{p}, u(\mathbf{x}_k))$. The gradient of $e(\ )$ wrt $\mathbf{p}$ is given by

(6.1) $$\nabla_{\mathbf{p}} e(\mathbf{p}, \mathbf{u}) = \nabla_{\mathbf{p}} \sum e_k(\mathbf{p}, u_k) = \sum \mathbf{x}_k = \mathbf{x}.$$

It would be nice if we could have an analogous gradient wrt $\mathbf{X}$ of the form

(6.2) $$\nabla_{\mathbf{x}} e(\mathbf{p}, \mathbf{u}(\mathbf{X})) = \mathbf{p}.$$

This seems at first sight nonsensical since $\mathbf{x}$ is not an argument of $e(\ )$. The expression would make sense if we could show that

(6.3) $$\nabla_{\mathbf{x}} e\big(\mathbf{p}, \mathbf{u}(\mathbf{X})\big)\Delta\mathbf{x} = \mathbf{p}\Delta\mathbf{x}$$

because (6.2) could then be viewed as an instruction to compute $\Delta e(\ )$ according to the formula

(6.4) $$e(\mathbf{p}, \mathbf{u}(\mathbf{X}^1)) - e(\mathbf{p}, \mathbf{u}(\mathbf{X}^0)) = \mathbf{p}\Delta\mathbf{x} + O_2(\Delta\mathbf{X}).$$

The validity of (6.4) follows from

(6.5) $$\sum \nabla_{\mathbf{x}_k} e_k\big(\mathbf{p}, u_k(\mathbf{x}_k)\big)\Delta\mathbf{x}_k = \sum \mathbf{p}\Delta\mathbf{x}_k = \mathbf{p}\Delta\mathbf{x}$$

The derivation is based on Balk's lemma (4.7) and the assumption that all households face the same price vector. The interpretation of (6.2) is that, when the variations of the $\Delta\mathbf{x}_k$ are small and their sum is given, their distribution is immaterial for the determination of $\nabla_{\mathbf{x}} e(\ )$. An alternative derivation of (6.2) is to regard it as an implication of (6.1), given duality.

With these preliminaries, we are in a position to define the logarithmic differentials of the *Aggregate Cost-of-Living C* and of *Aggregate Real Consumption R*. Using a vector notation

(6.6) $$d\ln C = \frac{\nabla_{\mathbf{p}} e(\mathbf{p}, \mathbf{u})\, d\,\mathbf{p}}{e(\mathbf{p}, \mathbf{u})} = \frac{\mathbf{x}\, d\,\mathbf{p}}{y} = \boldsymbol{\sigma}\, d\,\mathbf{ln}\,\mathbf{p}$$

$$(6.7) \qquad d\ln R = \frac{\nabla_{\mathbf{x}}\, e(\mathbf{p},\mathbf{u}(\mathbf{X}))\, d\,\mathbf{x}}{e(\mathbf{p},\mathbf{u}(\mathbf{X}))} = \frac{\mathbf{p}\, d\,\mathbf{x}}{y} = \boldsymbol{\sigma}\, d\,\mathbf{ln\, x}.$$

The differentials are those of Divisia integrals, this time defined on the vectors of aggregate consumption quantities and their prices. The appropriate indexes therefore again have the Törnqvist form.

7. AGGREGATION OF TÖRNQVIST INDEXES

The summation of proportional changes along an interval generally requires an integral, since the shares that serve as weights vary continuously. It is a remarkable property of Törnqvist indexes that they can be aggregated exactly, using only the initial and final shares. For this purpose, the Törnqvist price index is written as the product of a geometric Laspeyres price index and a geometric Paasche price index.

$$(7.1) \qquad \begin{aligned} P_T &= \left[\prod_i \left(\frac{p_i^1}{p_i^0}\right)^{s_i^0}\right]^{\frac{1}{2}} \left[\prod_i \left(\frac{p_i^1}{p_i^0}\right)^{s_i^1}\right]^{\frac{1}{2}} \\ &= \left(P_G^0\right)^{\frac{1}{2}} \left(P_G^1\right)^{\frac{1}{2}}. \end{aligned}$$

Using again the definitions of (5.1)

$$(7.2) \qquad \begin{aligned} P_{Tk} &= \left[\prod_{i\in k} \left(\frac{p_i^1}{p_i^0}\right)^{s_{ik}^0}\right]^{\frac{1}{2}} \left[\prod_{i\in k} \left(\frac{p_i^1}{p_i^0}\right)^{s_{ik}^1}\right]^{\frac{1}{2}} \\ &= \left(P_{Gk}^0\right)^{\frac{1}{2}} \left(P_{Gk}^1\right)^{\frac{1}{2}}. \end{aligned}$$

The aggregating equation is

$$(7.3) \qquad P_T = \prod_k \left[\left(P_{Gk}^0\right)^{\sigma_k^0} \left(P_{Gk}^1\right)^{\sigma_k^1}\right]^{\frac{1}{2}}.$$

Similarly,

$$(7.4) \qquad Q_T = \prod_k \left[\left(Q_{Gk}^0\right)^{\sigma_k^0} \left(Q_{Gk}^1\right)^{\sigma_k^1}\right]^{\frac{1}{2}}.$$

Unlike the literature on approximate aggregation of indexes, this aggregation is exact. In addition to its theoretical interest, it can also be used for efficient computation, for example in the context of the NIPAs. Once a set of indexes have been computed at a given level of aggregation, the raw data used for these computations is no longer required in computing the indexes of the next higher level.

8. CHAINING

Thus far, we analyzed bilateral comparisons based on the implicit assumption that the price and quantity vectors being compared are not too different, so that a reasonable approximation of the empirical to the theoretical measures will result. In a time series context, a bilateral index is suitable for year-to-year comparisons. It has long been recognized that a fixed index base cannot be maintained for too long, because as the changes in the variables become large the accuracy of the quadratic, or any other, approximation declines sharply. The alternative is some form of chaining. It has also been recognized that chaining introduces path dependence, usually referred to as violation of Fisher's circularity axiom. This has left practitioners in a quandary. The past practice in the context of the NIPAs has been to keep the base constant for 5 or 10 years and then to do some kind of rebasing to establish comparability of the different segments. The problems involved in this will be discussed further in the next section. Currently opinion has shifted towards the use of annually chained indexes. Furthermore, the view, at least of theoreticians, is that a symmetric index, not the usual Laspeyres formula should be used.

From a theoretical point of view, a chain index may be regarded as an approximation to a Divisia index over the entire interval. If year-to-year Törnqvist indexes are used, a sequence of quadratic approximations to the continuous path is obtained. From a numerical point of view, using more points of interpolation and thus more information, increases accuracy. In the present context, a limit to this improvement is set by annual data. Quarterly or monthly data introduce additional drift due to seasonal fluctuations. In addition, the accuracy of the data declines sharply. At the other end, the traditional method of holding the base constant over longer periods is pointless. The underlying continuous index is not changed thereby, only the approximation to it is worse.

For completeness, I state here how chain indexes can be used to compute levels. This is done by means of the usual convention that identifies the initial real magnitude with the nominal expenditure. The implied initial price level is 1. Let P^t be the price level and Q^t the real expenditure level, both at time t. In terms of these levels,

$$P^t = (1)\left(\frac{P^1}{P^0}\right)\cdots\left(\frac{P^t}{P^{t-1}}\right), \quad Q^t = \left(y^0\right)\left(\frac{Q^1}{Q^0}\right)\cdots\left(\frac{Q^t}{Q^{t-1}}\right). \tag{8.1}$$

In practice the ratios would be computed as suitable price and quantity indexes. The theory of the present paper suggests that these should be Törnqvist indexes.

9. THE ACCOUNTS OF SOCIETY

9.1. What is the Problem

In Section 2.6 I argued that NIPA statisticians have neither found a satisfactory method for computing the accounts in real terms, nor have they achieved agreement among themselves in this regard. In this section I argue that the solution is actually quite simple.

The very limited amount of discussion with regard to this issue that has taken place is virtually devoid of economic content. It is my aim to supply this content. The essence of economic analysis is substitution: efficiency requires the rates of substitution in consumption and production to be inversely proportional to market prices. If real magnitudes are defined in such a way that they do not satisfy this condition they have no economic meaning. This condition can only be met if real magnitudes have the same relative values as their nominal equivalents. This in turn implies that all values, or equivalently all prices, must be deflated with the same deflator.

Why are NIPA statisticians opposed to this simple method? I never heard a convincing answer, but my guess is the following: There is a wide spread belief that a deflated value is in the nature of an aggregated quantity and should behave like a quantity. The term ‘quantity index’ reflects that belief as does the use of such indexes as inputs to aggregate production or utility functions. A further belief is that this ‘quantity’ must be computed by a quantity index. Therefore GDP and its components are usually all computed directly by applying a quantity index to the corresponding nominal data.

9.2. Which Deflator

The theory of this paper indicates that the deflator should be a Törnqvist price index. The next question is what the index should be defined on. NIPA statisticians and economists generally assume that the GDP deflator should reflect the prices of all of its components. There is a substantial theoretical literature that disagrees. This literature began with Weitzman (1976). A comprehensive recent treatment is Sefton and Weale (2004). At the center of this literature is the definition of net national income (NNI). Two definitions are offered. At the level of the individual consumer these are: **a.** His expected, discounted future stream of real consumption. **b.** That current level of real consumption that can be indefinitely maintained. They show that both measures are equivalent. The aggregate definitions are the sums of these measures over all consumers. The fundamental result is that the two definitions are equivalent at öthe aggregate level also and that the NNI can be measured as the NNP *deflated by a consumer price index* (CPI). Furthermore, the relevant theoretical index turns out to be the Divisia price index. The theoretical literature thus comes to conclusions that are analogous to those of this paper.

That NNP deflated by the CPI is the appropriate aggregate welfare measure also has a simple intuitive interpretation: If the entire NNP were devoted to consumption, then by definition, this level could just be maintained and to measure it in real terms, the CPI is evidently the appropriate measure.

There is also a pragmatic reason for choosing the consumption deflator. Production technologies change so radically over time that in my opinion a meaningful index for capital goods cannot be constructed. Statisticians deal with this problem by taking capital goods that cost the same as being equivalent. This is not economically meaningful since it ignores the technological progress. Serious measurement problems are also present in relation to the government and foreign sectors.

9.3. Further Issues

There are further problematic aspects regarding the current definitions of various aggregate product and income statistics. The definitions are to some extent untenable from a theoretical point of view and have pathological consequences. To give just one example: If the only change is a reduction of import prices, the GDP deflator as currently constructed will rise! I have pursued some of these issues further in Hillinger (2002/2003). A number of such anomalies are discussed by Rakowski (1999). He also conducted a survey showing that prominent economists react with utter confusion when confronted with such anomalies.

10. AN INTEGRATED SYSTEM OF ACCOUNTS FOR MEASURING INFLATION

In the preceding section I argued for a single deflator to obtain a consistent NIPA in real terms. Such an account is needed for the purpose of macroeconomic analysis and model construction. There is also a need for disaggregated price statistics. Presently such prices are quoted in an ad hoc fashion. Here I argue for a detailed accounting for prices in terms of sectors and subsectors.

In Section 8 I discussed the aggregation of Törnqvist indexes over component indexes. Such an aggregation process is particularly interesting for the Törnqvist price indexes and suggests the creation of a system of accounts showing how inflation at each higher level derives from the inflations of the components. At present the public discussion of inflation is concentrated on very few indexes: most importantly the CPI, to a lesser extent the index of producer prices and rarely the GDP deflator. The sectoral determination of these indexes is reported only episodically.

I believe that a set of three such accounts would be most informative. The first would show annual rates of inflation. From this table one could, for example, see how much of the CPI inflation of a given year, or quarter, was caused by each of its components. A second account would present the corresponding price levels, starting from a value of unity in some base year. This shows the cumulative amount of inflation and also allows a quick comparison of the price levels at any two periods. In a final account, all sectoral price levels would be 'deflated' by the general price level. This would be a table of 'relative prices'. If for some sector *k* and period *t* the table shows that $P_k^t = 2$, the implication is that, starting from the base period, prices in that sector increased twice as much as the average for the economy. A system of accounts for relative prices would be a genuine novelty and an increase in economically meaningful information

11. CONCLUDING REMARKS

The principal conclusions of the paper are: **a.** The Törnqvist index is the only one that can be integrated in a realistic and encompassing economic theory. **b.** The GDP deflator should be the CPI in the form of a chained Törnqvist price index. **c.** The theory of economic measurement should be a core subject for all economists. If economists are uninformed about both theoretical and practical aspects of the data they use, the scientific status of the discipline is in doubt.

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